Mid Semestral Exam
Algebra-II
MMath-I, 25-26

Time: 2 hours 30 mins

Max marks: 30

Answer any **six** *questions. No marks will be awarded in absence of complete justification. Notations are as used in class. All questions carry equal marks.*

1. Show that a group G contains exactly one proper non-trivial subgroup if and only if G is cyclic of order p^2 for some prime p.
2. Let G be a group of order $2m$, where m is odd. Prove that G contains a normal subgroup of order m.
3. Show that $\mathbb{Q}$ does not contain a subgroup of finite index.
4. Compute the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$.
5. Consider the subgroup $G = \{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & d \end{pmatrix}, ad \neq 0\}$ of $GL_3(\mathbb{R})$. Show that G is a semidirect product of the additive group $\mathbb{R}^2$ by $\mathbb{R}^\times \times \mathbb{R}^\times$. Is it a direct product of these groups?
6. Show that $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n}$ if and only if d divides n.
7. Define perfect fields. Show that a polynomial over a perfect field is separable if and only if it is the product of distinct irreducible polynomials.

****